Energy transport and chaos in a one-dimensional disordered nonlinear stub lattice

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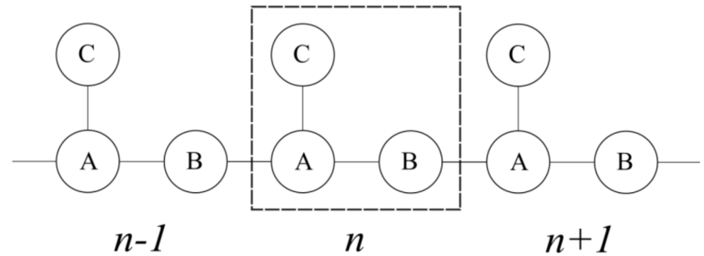
Outline

- **The one-dimensional disordered nonlinear stub lattice model**
  - ✓ **Linear system (flat frequency band, band gaps)**
  - ✓ **Different dynamical behaviors**
- **Numerical results**
  - ✓ **Representative cases of the various dynamical regimes**
  - ✓ **Global investigation of the parameter space**
  - ✓ **The effect of frequency gaps**
- **Summary**

The stub lattice model



Each one of the N unit cells contains three sites labeled A, B and C [Flach et al., EPL (2014); Luck, J. Phys. A (2019)].

$$H = \sum_{n=1}^{N} \left[\varepsilon_n^{(A)} \left|\psi_n^{(A)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(A)}\right|^4 + \varepsilon_n^{(B)} \left|\psi_n^{(B)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(B)}\right|^4 + \varepsilon_n^{(C)} \left|\psi_n^{(C)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(C)}\right|^4 \right.$$

$$\left. - \left(\psi_n^{(C)*}\psi_n^{(A)} + \psi_n^{(C)}\psi_n^{(A)*}\right) - \left(\psi_n^{(A)*}\psi_n^{(B)} + \psi_n^{(A)}\psi_n^{(B)*}\right) - \left(\psi_{n+1}^{(A)*}\psi_n^{(B)} + \psi_{n+1}^{(A)}\psi_n^{(B)*}\right) \right]$$

with fixed boundary conditions $\boldsymbol{\psi_{N+1}^{(A)} = 0}$.

$\boldsymbol{\beta}$: nonlinearity parameter, W: disorder strength. H: total energy.

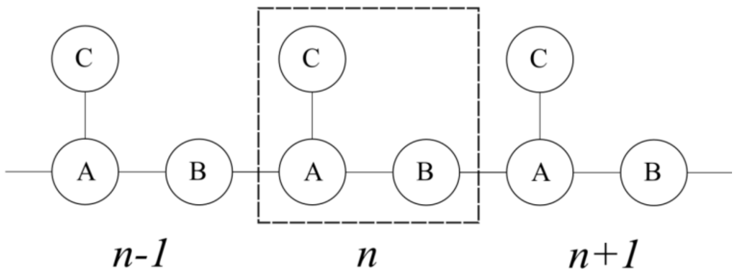
$\boldsymbol{\varepsilon_n^{(K)}}$ (K stands for A, B or C) chosen uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$.

The stub lattice Hamiltonian has formal similarities to the Hamiltonian of the discrete nonlinear Schrödinger equation in one spatial dimension (1D DDNLS):

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - (\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l)$$

The stub lattice model

The canonical transformation $\psi_n^{(K)} = \frac{q_n^{(K)} + i p_n^{(K)}}{\sqrt{2}}$ gives the Hamiltonian the form



$$H = \sum_{n=1}^{N} \left\{ \frac{\varepsilon_n^{(A)}}{2} \left[\left(q_n^{(A)}\right)^2 + \left(p_n^{(A)}\right)^2 \right] + \frac{\beta}{8} \left[\left(q_n^{(A)}\right)^2 + \left(p_n^{(A)}\right)^2 \right]^2 + \frac{\varepsilon_n^{(B)}}{2} \left[\left(q_n^{(B)}\right)^2 + \left(p_n^{(B)}\right)^2 \right] \right.$$
$$+ \frac{\beta}{8} \left[\left(q_n^{(B)}\right)^2 + \left(p_n^{(B)}\right)^2 \right]^2 + \frac{\varepsilon_n^{(C)}}{2} \left[\left(q_n^{(C)}\right)^2 + \left(p_n^{(C)}\right)^2 \right] + \frac{\beta}{8} \left[\left(q_n^{(C)}\right)^2 + \left(p_n^{(C)}\right)^2 \right]^2$$
$$\left. - \left(p_n^{(C)} p_n^{(A)} + q_n^{(C)} q_n^{(A)} \right) - \left(p_n^{(A)} p_n^{(B)} + q_n^{(A)} q_n^{(B)} \right) - \left(p_{n+1}^{(A)} p_n^{(B)} + q_{n+1}^{(A)} q_n^{(B)} \right) \right\}$$

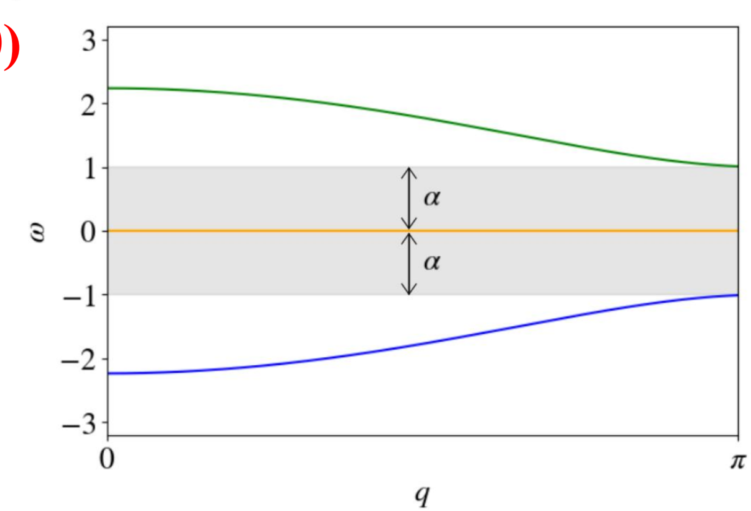
Conserved quantities:

The total energy *H*, and the total norm *S* of the wave packet:

$$S = \sum_{n=1}^{N} \sum_{K} \frac{1}{2} \left[\left(q_n^{(K)}\right)^2 + \left(p_n^{(K)}\right)^2 \right]$$

The linear system

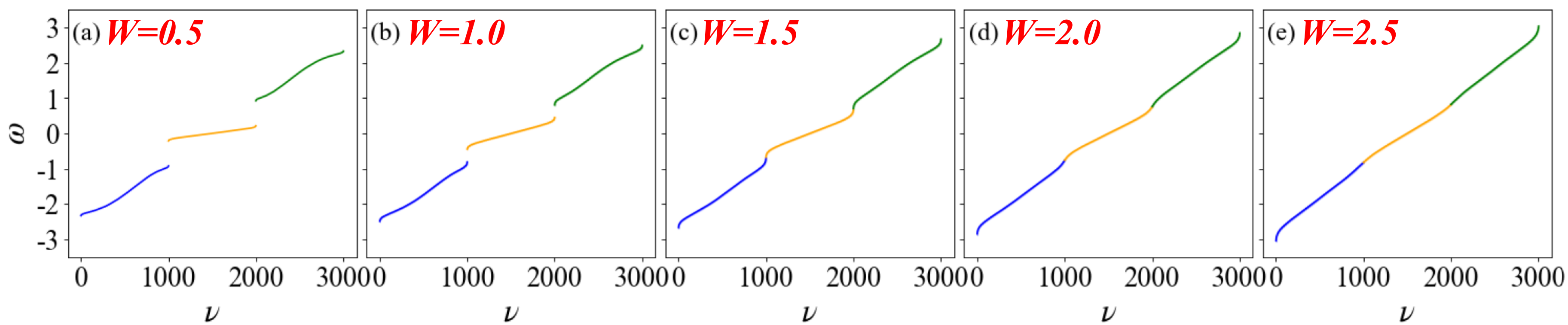
Considering the linear ($\beta = 0$), ordered ($\varepsilon_n^{(K)}$=0) stub lattice model and the wave function $\psi_N^{(K)} = U^{(K)} e^{-iqn}\, e^{-i\omega t}$ we obtain the system's dispersion relation which consists of 3 distinct bands (gap size $\alpha = 1$):



$$\omega = \sqrt{3 + 2\ \cos q},\ \omega = 0 \text{ (flat band)},$$

$$\omega = -\sqrt{3 + 2\ \cos q}.$$

Introducing disorder, the gap disappears for $W \approx 1.58$. Frequencies (ω) vs. mode number (ν) for $N = 1000$ (averaged over 50 disorder realizations):



Width of frequency spectrum:

$$\Delta = \omega_{max} - \omega_{min} = \left(\frac{W}{2} + \sqrt{5}\right) - \left(-\frac{W}{2} - \sqrt{5}\right) = W + 2\sqrt{5}$$

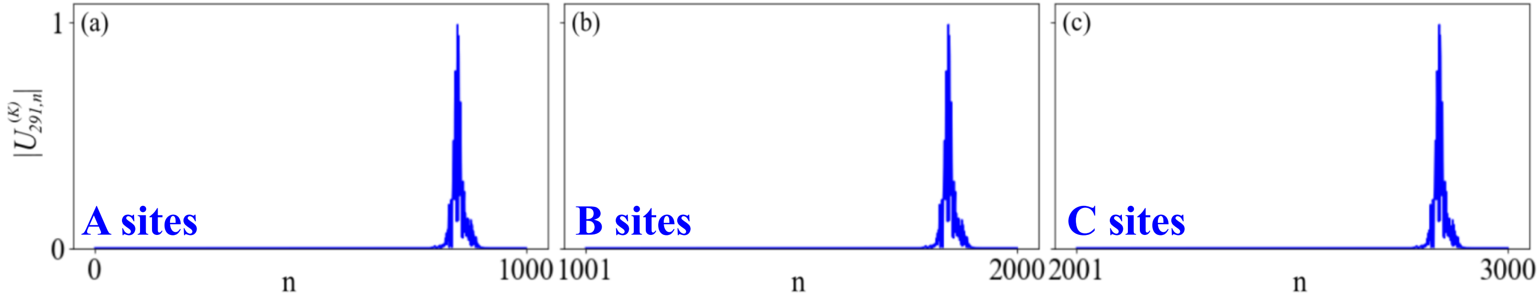
The linear system

Localization volume $V_\nu^{(i)}$ of normal modes (NMs) in ranges $i = 1, 2, 3$ (A, B, C sites respectively):

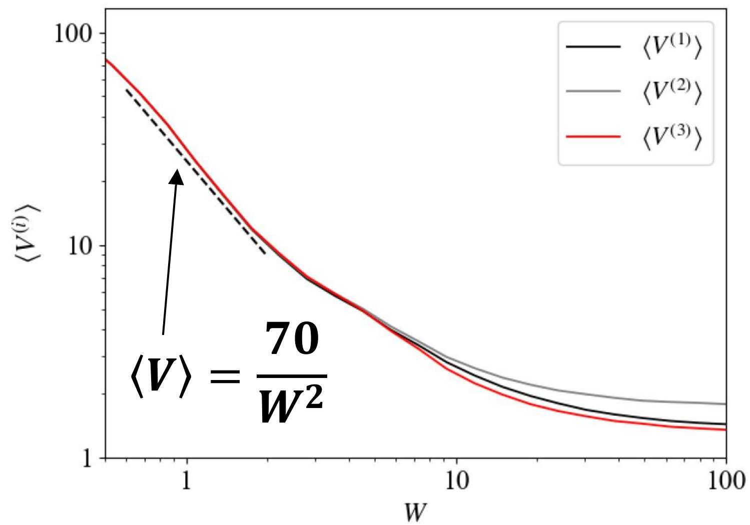
$$V_\nu^{(i)} = \sqrt{12 m_2^{(\nu,i)}} + 1$$

where $m_2^{(\nu,i)} = \sum_{n=(i-1)N+1}^{iN} \left(n - \bar{n}_\nu^{(i)}\right)^2 \left|U_{\nu,n}^{(i)}\right|^2$ **and** $\bar{n}_\nu^{(i)} = \sum_{n=(i-1)N+1}^{iN} n \left|U_{\nu,n}^{(i)}\right|^2$

Mode $\nu = 291$ for $W = 1$ ($\beta = 0$).



Considering modes whose mean position is at the central one-third of the 3 ranges, we compute the average (over disorder realizations) $\langle V^{(i)} \rangle$ and then the



average localization volume of NMs

$$\langle V \rangle = \max_i \langle V^{(i)} \rangle, \quad i = 1, 2, 3.$$

Computational techniques

Norm distribution: $\xi_n = \sum_K \frac{\left(q_n^{(K)}\right)^2 + \left(p_n^{(K)}\right)^2}{2S}$

Second moment: $m_2 = \sum_{n=1}^{N} (n - \bar{n})\xi_n$ **with** $\bar{n} = \sum_{n=1}^{N} n\xi_n$

Participation number: $P = \frac{1}{\sum_{n=1}^{N} \xi_n^2}$

Finite-time maximum Lyapunov exponent (ftMLE): $\Lambda(t) = \frac{1}{t} \ln\left(\frac{||\mathbf{w}(t)||}{||\mathbf{w}(0)||}\right)$

Deviation vector distribution (DVD):

$$\xi_n^D(t) = \frac{\sum_K \left[\left(\delta q_n^{(K)}(t)\right)^2 + \left(\delta p_n^{(K)}(t)\right)^2\right]}{\sum_{n=1}^{N} \sum_K \left[\left(\delta q_n^{(K)}(t)\right)^2 + \left(\delta p_n^{(K)}(t)\right)^2\right]}$$

Different Dynamical Regimes

1D DDNLS: Three dynamical regimes [Flach et al., PRL (2009); S. et al., PRE (2009); Flach, Chem. Phys (2010); Laptyeva et al., EPL (2010); Bodyfelt et al., PRE (2011); S. et al., PRL (2013); Senyange et al., PRE (2018)]

Δ: width of the frequency spectrum, $d = \frac{\Delta}{\langle V \rangle}$: average spacing of interacting modes,

$\delta = \beta s$: nonlinear frequency shift.

Weak chaos regime: $\delta < d, m_2 \sim t^{1/3}, P \sim t^{1/6}, \Lambda \sim t^{-0.25}$. NMs are weakly interacting.

Strong chaos regime: $d < \delta < 2, m_2 \sim t^{1/2}, P \sim t^{1/4}, \Lambda \sim t^{-0.3}$. Almost all NMs in the packet are resonantly interacting.

Selftrapping Regime: $\delta > 2$. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, while a small part of the wave packet subdiffuses.

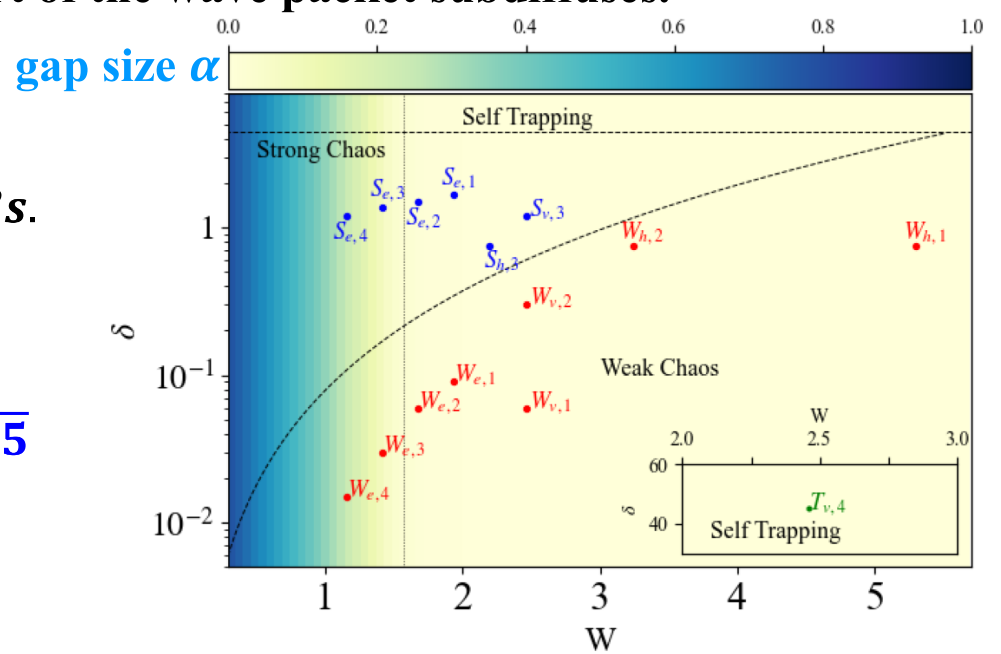
Stub lattice:

$$\Delta = W + 2\sqrt{5},\ d = \frac{\Delta}{\langle V \rangle} = \frac{W^2(W+2\sqrt{5})}{70},\ \delta = \beta s.$$

Weak chaos regime: $\delta < \frac{W^2(W+2\sqrt{5})}{70}$

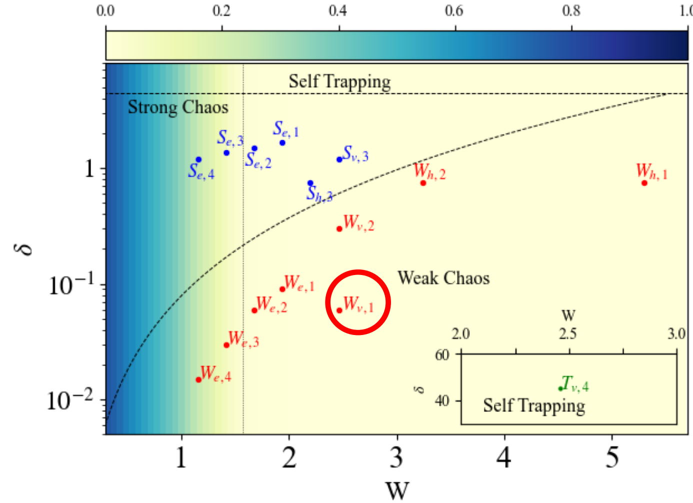
Strong chaos regime: $\frac{W^2(W+2\sqrt{5})}{70} < \delta < 2\sqrt{5}$

Selftrapping Regime: $\delta > 2\sqrt{5}$



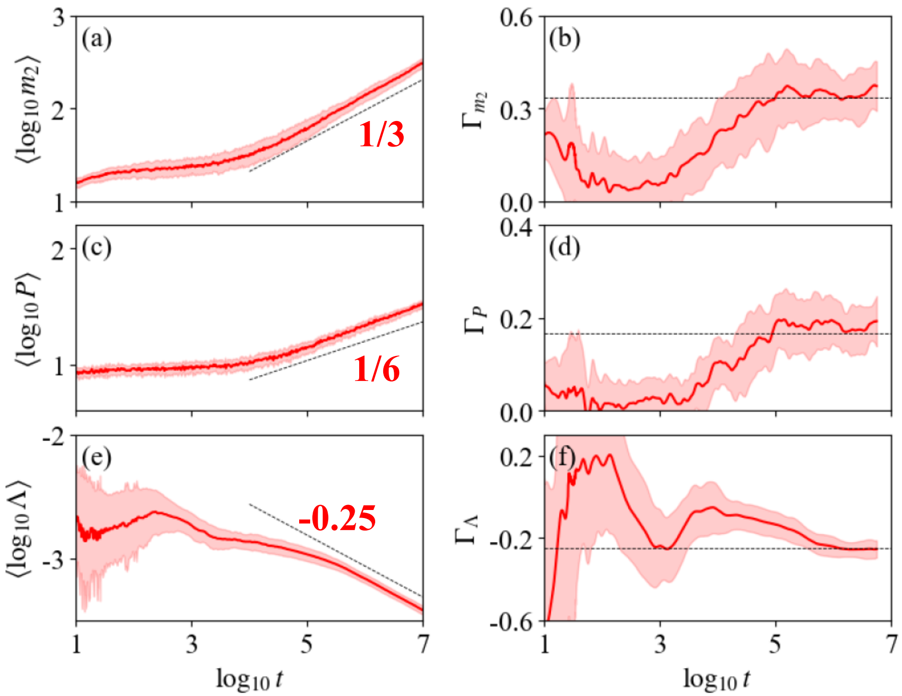
Weak chaos (representative case)

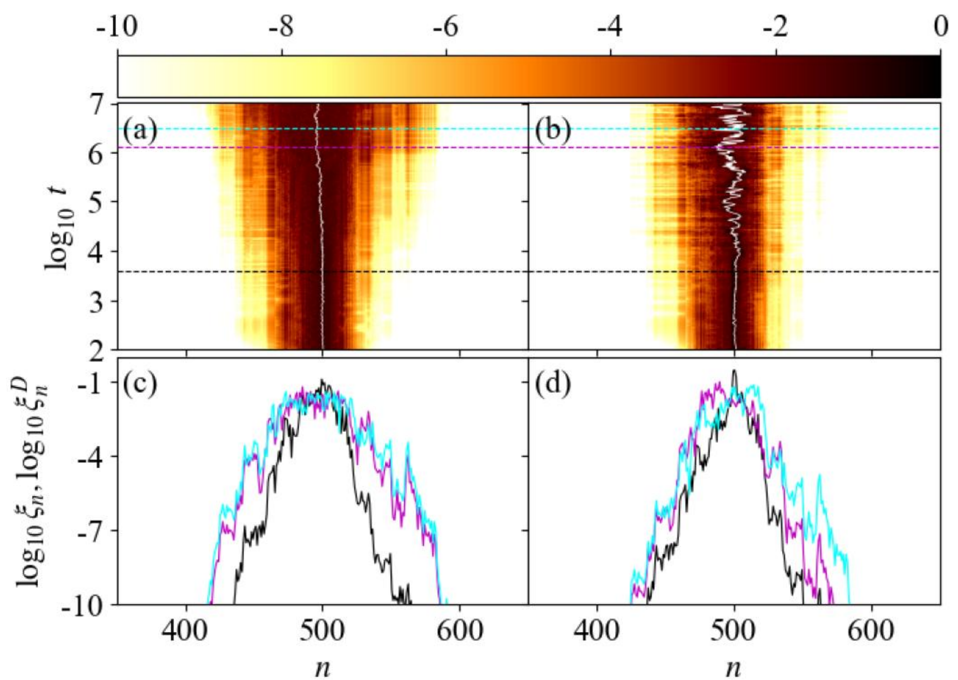
$\boldsymbol{\beta = 0.02, W = 2.46, L = 12, N = 1001}$



Typically, we present the time evolution of a quantity $\langle R \rangle$ in log-log scale and numerically estimate the related rate of change as:

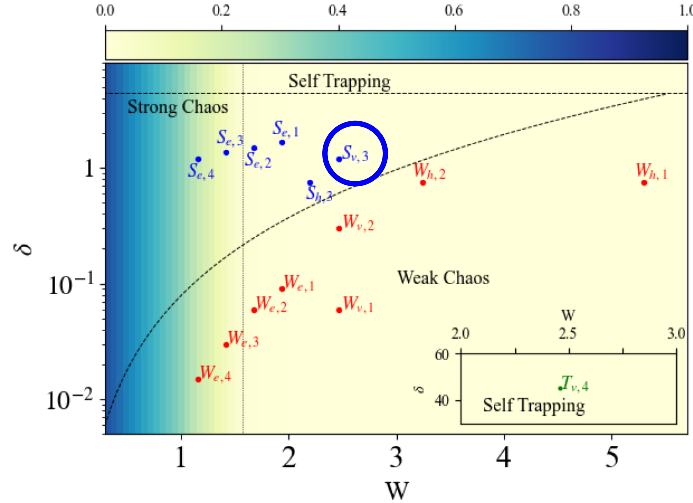
$$\Gamma_R(\log_{10} t) = \frac{d\langle \log_{10} R \rangle}{d \log_{10} t}$$

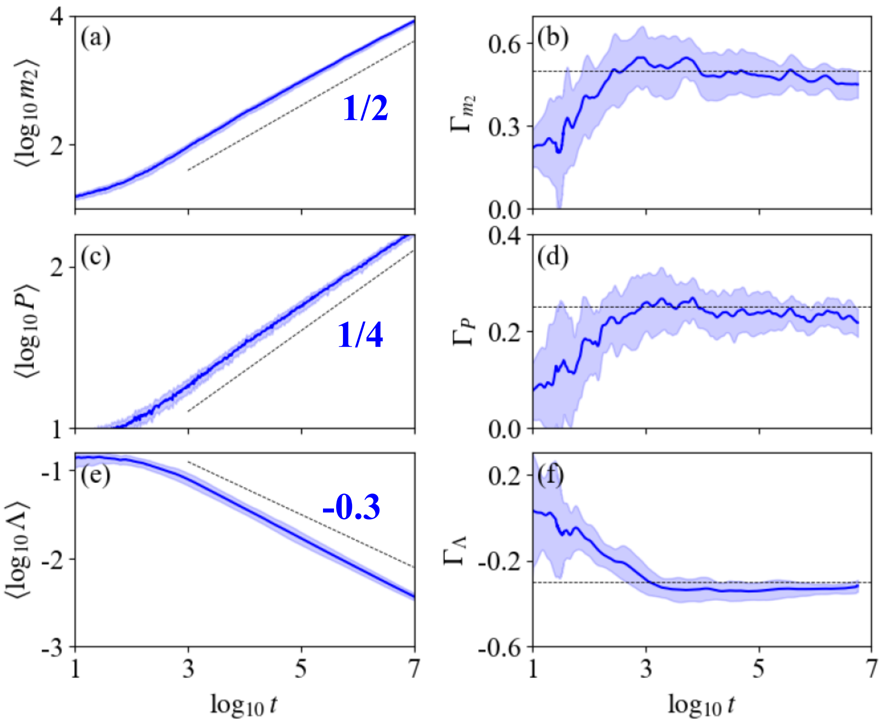


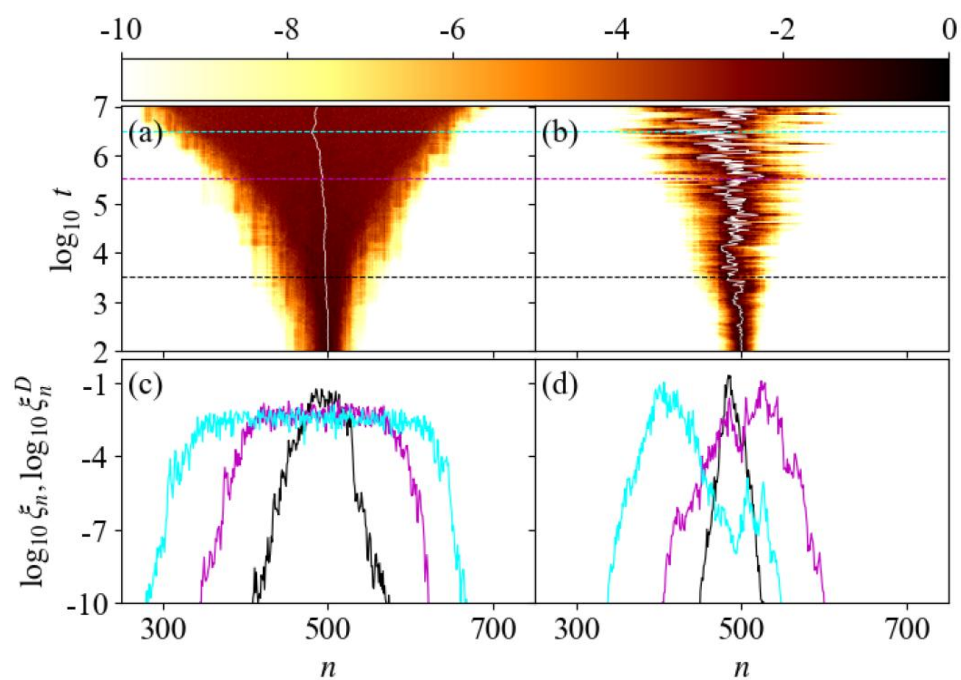


Strong chaos (representative case)

$\boldsymbol{\beta = 0.4, W = 2.46, L = 12, N = 1001}$

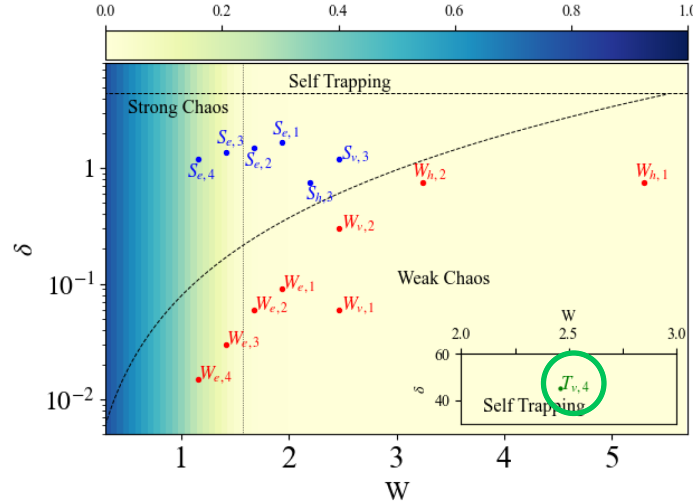


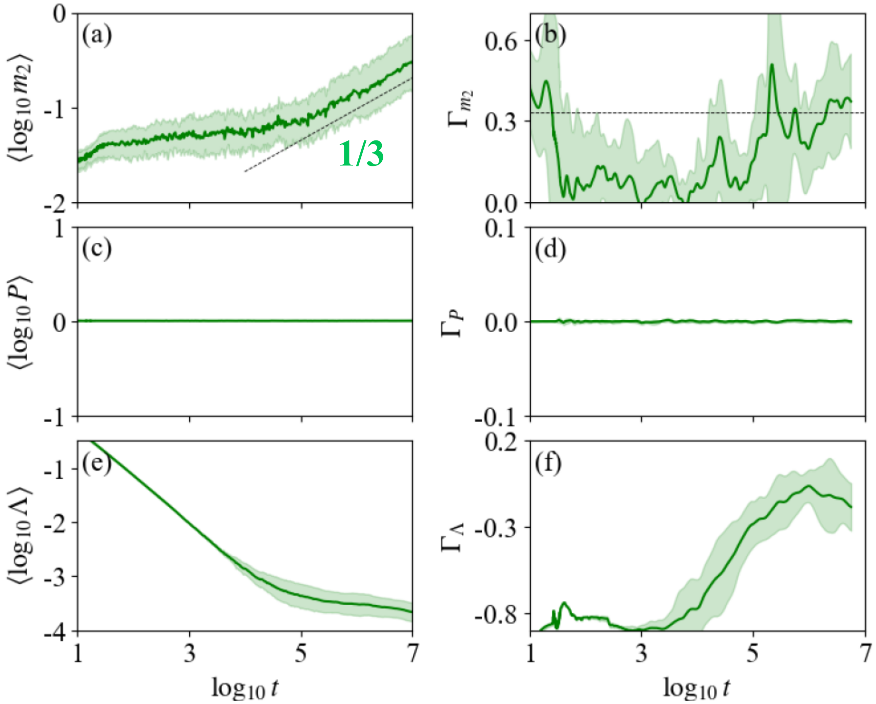


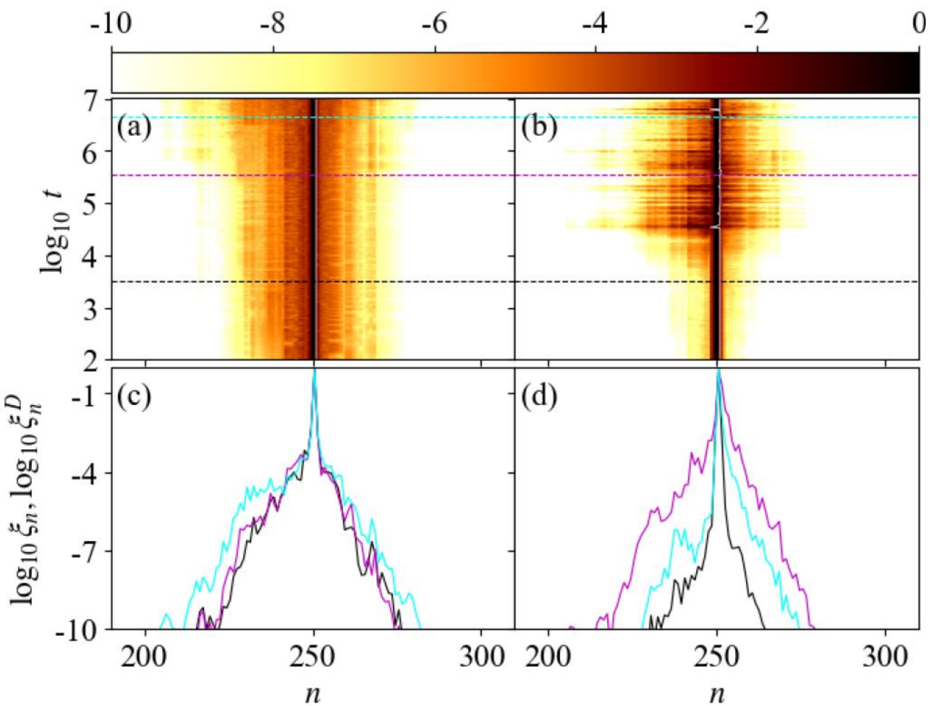


Selftrapping (representative case)

$\beta = 15, W = 2.46, L = 1, N = 501$





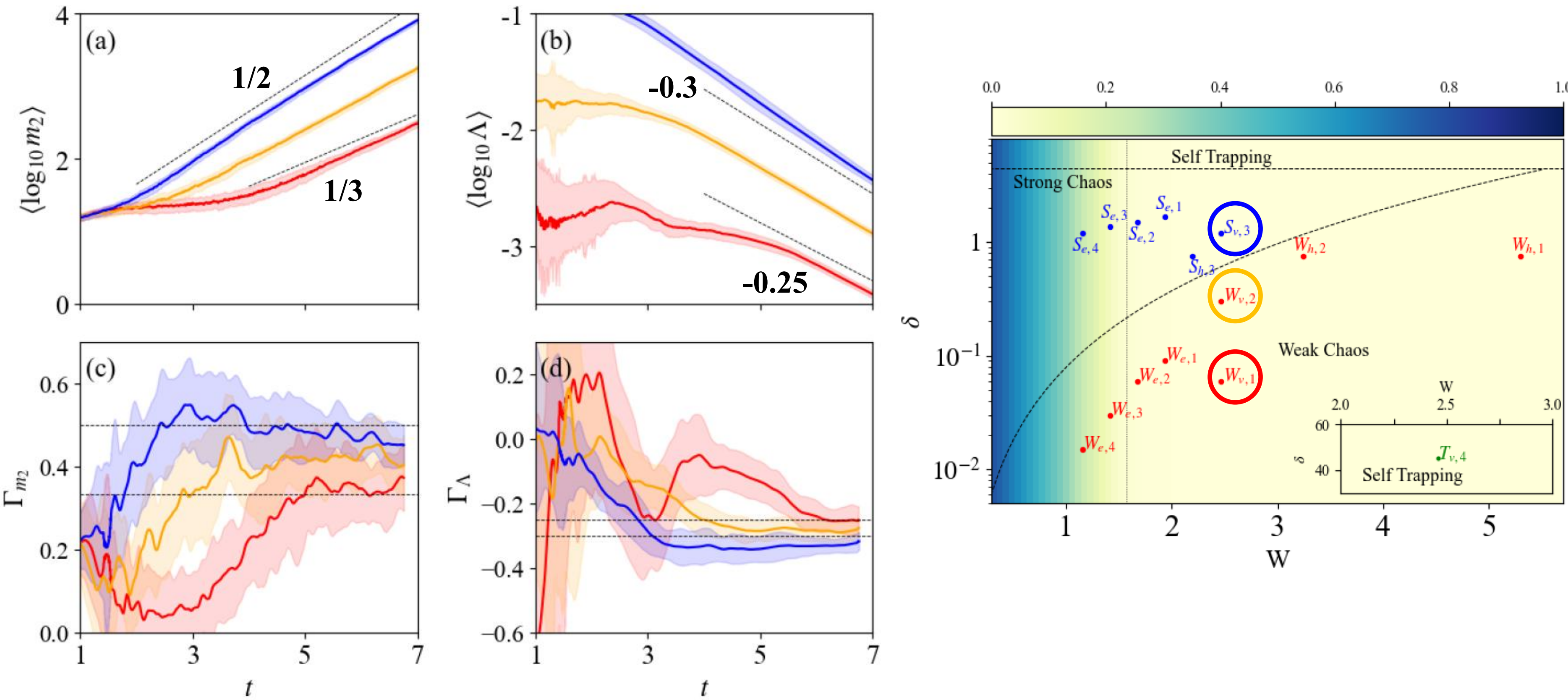


Global investigation I

$\boldsymbol{\beta = 0.02, W = 2.46, L = 12, N = 1001}$

$\boldsymbol{\beta = 0.045, W = 2.46, L = 12, N = 1001}$

$\boldsymbol{\beta = 0.4, W = 2.46, L = 12, N = 1001}$

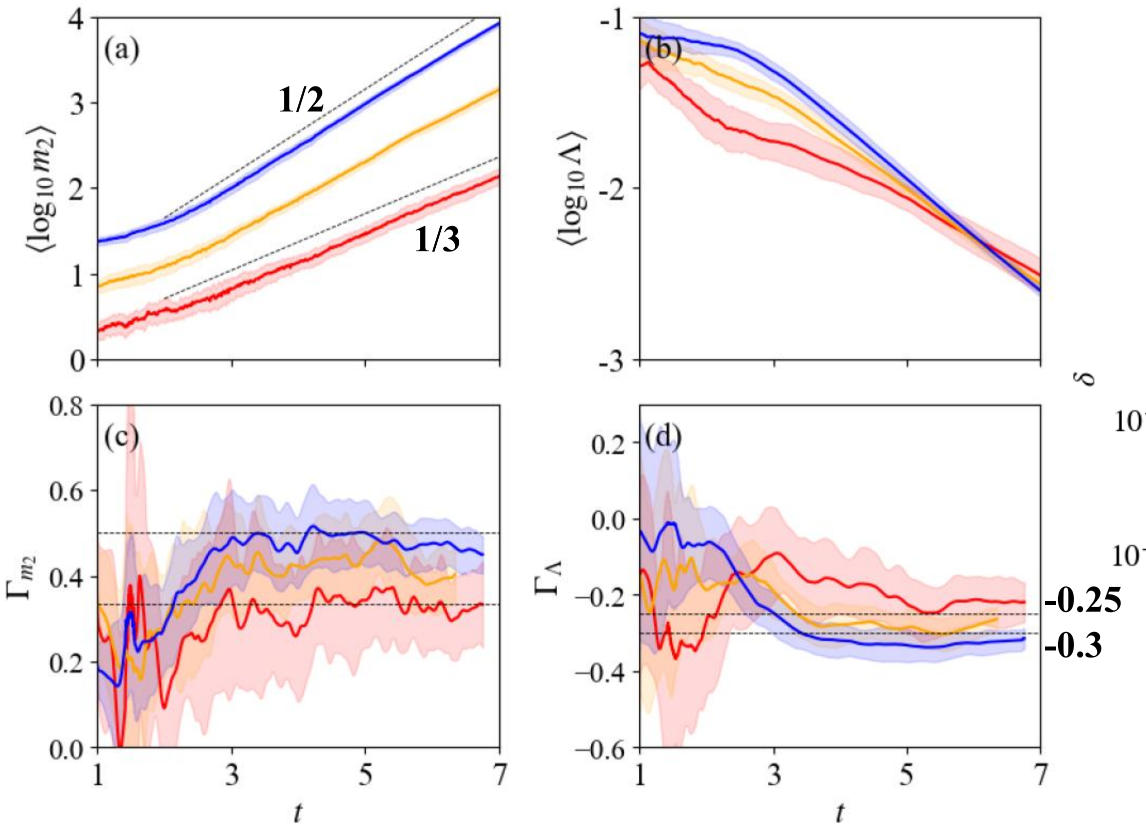


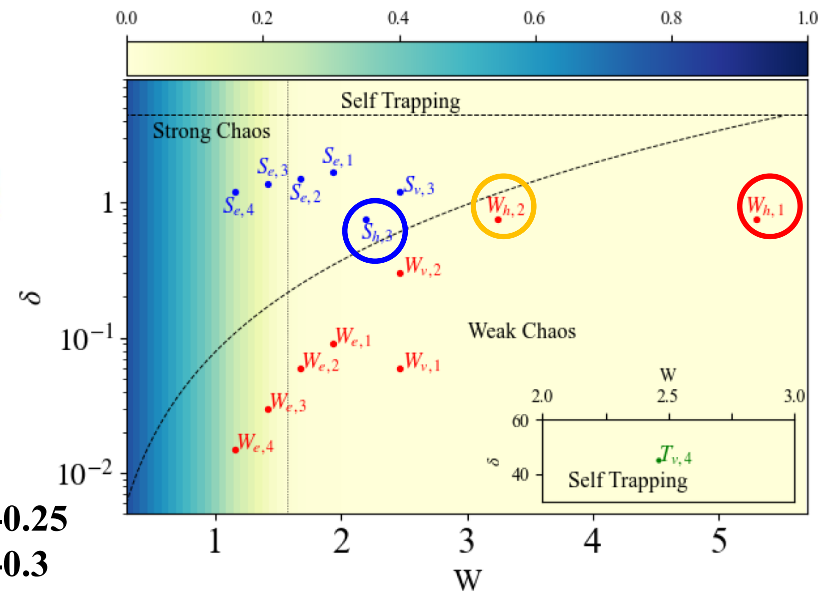
Global investigation II

$\boldsymbol{\beta = 0.25, W = 5.2, L = 3, N = 1001}$

$\boldsymbol{\beta = 0.25, W = 3.5, L = 6, N = 1001}$

$\boldsymbol{\beta = 0.25, W = 2.2, L = 15, N = 1001}$





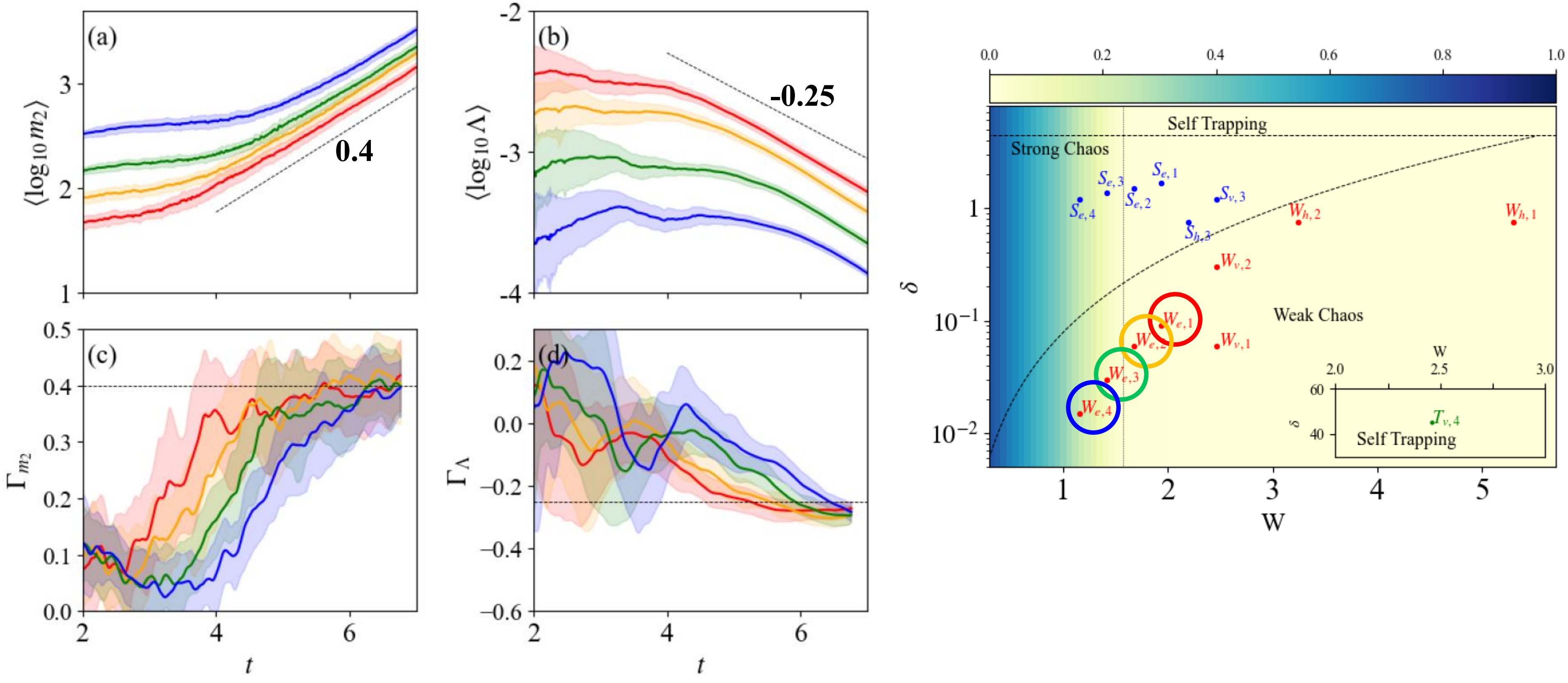
Effect of frequency gaps (Weak chaos)

$\beta = 0.03, W = 1.94, L = 18, N = 1001, \alpha = 0$

$\beta = 0.02, W = 1.68, L = 24, N = 1001, \alpha = 0$

$\beta = 0.01, W = 1.42, L = 35, N = 1001, \alpha = 0.007$

$\beta = 0.005, W = 1.16, L = 53, N = 1001, \alpha = 0.124$



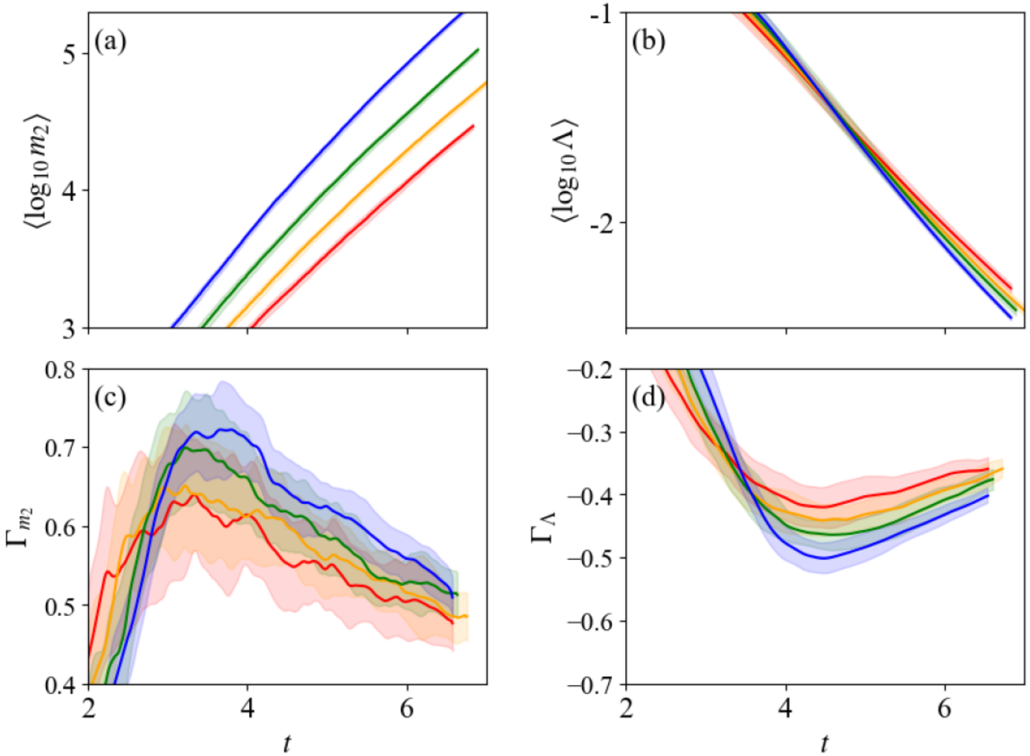
Effect of frequency gaps (Strong chaos)

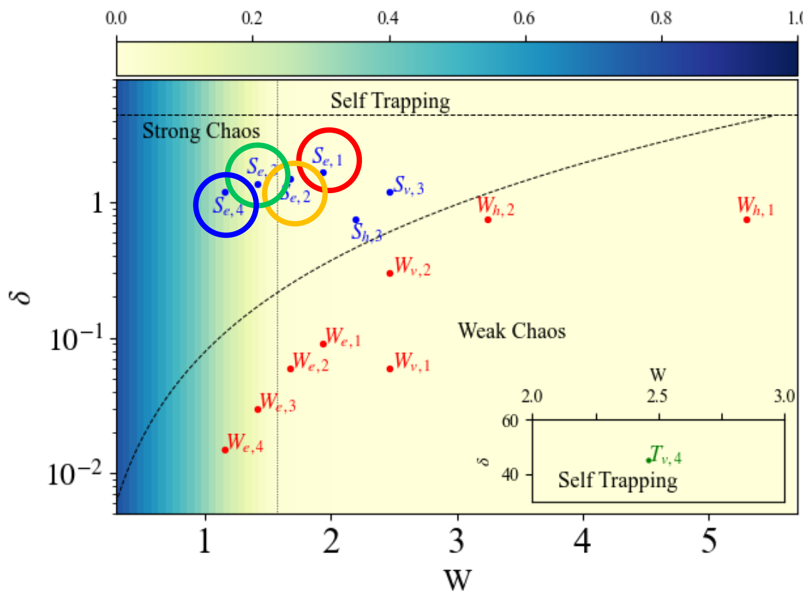
$\beta = 0.55, W = 1.94, L = 18, N = 1001, \alpha = 0$

$\beta = 0.5, W = 1.68, L = 24, N = 1001, \alpha = 0$

$\beta = 0.45, W = 1.42, L = 35, N = 1001, \alpha = 0.007$

$\beta = 0.4, W = 1.16, L = 53, N = 1001, \alpha = 0.124$





Summary

We investigated in detail the spatiotemporal chaotic behavior of the one-dimensional disordered nonlinear stub lattice model, whose linear counterpart exhibits a flat frequency band.

- **Identification of 2 different dynamical spreading regimes:**
  - **weak chaos regime: $m_2 \sim t^{1/3}, P \sim t^{1/6}, \Lambda \sim t^{-0.25}$**
  - **strong chaos regime: $m_2 \sim t^{1/2}, P \sim t^{1/4}, \Lambda \sim t^{-0.3}$**
- **The transition between the different dynamical regimes is not abrupt but happens in a rather smooth fashion.**
- **The presence of the frequency gap does not seem to strongly affect the dynamics of the weak chaos regime, while in the case of strong chaos further investigations are needed, as our results are rather inconclusive.**
- **The ftMLE reveals the decrease of the system's chaoticity in time.**
- **The DVDs provide information about the propagation of chaos.**

  Wandering of localized chaotic hot spots in the lattice's excited part homogenize chaos.

Main references

- Bodyfelt, Laptyeva, S., Krimer, Flach (2011) PRE, 84, 016205
- Flach, Krimer, S. (2009) PRL, 102, 024101
- Flach (2010) Chem. Phys., 375, 548
- Flach, Leykam, Bodyfelt, Matthies, Desyatnikov (2014) EPL, 105, 30001
- Laptyeva, Bodyfelt, Krimer, S., Flach (2010) EPL, 91, 30001
- Luck (2019) J. Phys. A, 52, 205301
- S., Krimer, Komineas, Flach (2009) PRE, 79, 056211
- S., Gkolias, Flach (2013) PRL, 111, 064101
- Senyange, Many Manda, S. (2018) PRE, 98, 052229

International Workshop, 2-6 February 2026
Cape Town, South Africa

WILHELM UND ELSE HERAEUS-STIFTUNG

Wilhelm und Else Heraeus-Stiftung
University of Cape Town (UCT)

South African-German WE-Heraeus Seminar:
Nonlinear dynamics and anomalous transport

International Workshop
2 - 6 February 2026
Cape Town, South Africa

The study of unconventional dynamics and transport in low dimension has a long and distinguished history. It is thus perhaps surprising how much progress this field has seen in recent years, both in theory and experiment. This event is devoted to covering these developments in depth, starting from an introduction to the basics of the field and leading up to the cutting edge of current day research. The goal of this event is to bring together different communities which have studied anomalous dynamics and transport in low dimensions.

The format is chosen to maximise accessibility to a diverse set of participants. It is particularly aimed at researchers in the south of Africa, with an emphasis of career stages up to junior faculty. We aim to enable contacts between researchers based in Africa and in Germany at similar career stages, in the hope that these will persist as the researchers move ahead in their careers.

Topics include:

- Anomalous dynamics and transport
- Many-body dynamics
- Nonlinear and chaotic dynamics
- Nonlinear lattices
- Quantum transport
- Thermalization
- Non-Hermitian dynamics
- Large deviations
- Disordered systems

List of invited participants

Vassos Achilleos (FR)
Monika Aidelsburger* (DE)
Nora Alexeeva (SA)
Igor Barashenkov (SA)
André Botha (SA)
Pieter Claeys (DE)
Fabian Essler (UK)
Flore Kunst (DE)
Stefano Lepri (IT)
Joel Moore (US)
Silvia Pappalardi (DE)
Tomaž Prosen (SI)
Georgios Theocharis (FR)
Hugo Touchette (SA)
Ruben Verresen (US)
Paul Woafo (CM)
** to be confirmed*

Scientific coordinators

Roderich Moessner
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Haris Skokos
(University of Cape Town)

Organisation

Kristina Aliabiev
(MPI-PKS Dresden)

Yvonne Brown
(University of Cape Town)

Applications received before 30 September 2025 are considered preferentially.

Applications are welcome and should be made by using the application form on the website of the event. Applications received before 30 September 2025 are considered preferentially. The number of attendees is limited.

For further information please contact:
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Topics include:

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The event is particularly aimed at researchers of career stages up to junior faculty.

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